647510 **MSC4MatC403x** **Seat No :________**

M.Sc.(Maths.) Semester - 4 (*CBCS*) Examination
April/May-2023 (NEW COURSE)
Number Theory -2 (Core)

Time: 2:30 Hours **Marks: 70**

Instructions:
1. All questions are compulsory.
2. Figures to the right indicate marks.

Q.1(A) Answer the following questions. (Any Two) (10)

1. If θ is an irrational number then $\exists$ infinitely many rational $\frac{a}{b}$ such that $\left|\theta - \frac{a}{b}\right| < \frac{1}{b^2}$; $b > 0$ and $(a, b) = 1$.
2. There is a polynomial $f_n(x)$ of degree n, leading coefficient 1 with integer coefficients such that $f_n(2\cos\theta) = 2\cos n\theta$.
3. If θ is an irrational number then there are integers $a_0, a_1, \dots, a_n, \dots$ such that $a_i \geq 1$; $\forall\, i = 1,2,3,4, \dots$ and $\theta = < a_0,\ a_1, \dots, a_{n,} \dots >$.

Q.1(B) Answer the following question. (Any One) (04)

1. If a polynomial equation with integral coefficients $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$, $a_n \neq 0$ has a non zero rational solution $\frac{a}{b}$ where the integers a and b are relatively prime then show that a/a_0 and b/a_n.
2. Suppose **(I)** $\lim\limits_{n\to\infty} r_{2n} = \theta$ **(II)** $\lim\limits_{n\to\infty} r_{2n-1} = \theta'$ then $\theta = \theta'$.

Q.2(A) Answer the following questions. (Any Two) (10)

1. If θ is an irrational number and suppose $\frac{a}{b}$ rational number such that $b > 0$, $(a, b) = 1$ and $\left|\theta - \frac{a}{b}\right| < \frac{1}{2b^2}$ then $\frac{a}{b} = \frac{h_n}{k_n}$, for some $n \geq 0$.
2. If θ is an irrational number then $\exists$ infinitely many rational $\frac{a}{b}$ such that $\left|\theta - \frac{a}{b}\right| < \frac{1}{\sqrt{5}b^2}$; $b >\ 0$ and $(a, b) = 1$.
3. Show that the value of any infinite continued fraction is an irrational number.

Q.2(B) Answer the following question. (Any One) (04)

1. Expand $\frac{\sqrt{5}+1}{2}$ as an infinite continued fraction.
2. Expand the rational fraction $29/51$ into finite simple continued fractions.

Q.3(A) Answer the following questions. (Any Two) (10)

1. If (x_1, y_1) is the smallest positive solution of $x^2 - dy^2 = 1$, d being a positive integer not a perfect square, then every positive solution are given by (x_n, y_n) for $n = 1, 2, 3, \dots$ where $x_n \; and \; y_n$ are the integers defined by $x_n + \sqrt{d}y_n = \left(x_1 + \sqrt{d}y_1\right)^n$.
2. Prove that π is an irrational number.
3. If θ is a quadratic irrational number then the continued fraction expansion of θ is purely periodic iff (1) $\theta > 1$, (2) $-1 < \theta' < 0$. Where θ' is conjugate of θ.

Q.3(B) Answer the following question. (Any One) **(04)**

1. Show that $\sqrt{2}$ is a quadratic irrational number.
2. If x is a real number and if $x + x^{-1} < \sqrt{5}$ then **(I)** $x < \frac{\sqrt{5}+1}{2}$ **(II)** $x^{-1} > \frac{\sqrt{5}-1}{2}$.

Q.4(A) Answer the following questions. (Any Two) **(10)**

1. Find all solution of $30x - 21y = 54$.
2. The value of $f(x) = x^4 + x^3 + x^2 + x + 1$ is perfect square number only for $x = -1, 0, 3$ and for some values of $x, f(x)$ is not perfect square number.
3. If u and v are $\geq 1, (u, v) = 1$ and if uv is perfect square number then u is perfect square number and v is also perfect square number.

Q.4(B) Answer the following question. (Any One) **(04)**

1. If x, y, z is primitive Pythagorean triple then show that one of the integers x *or* y is even while the other is odd.
2. **(I)** What is the Diophantine equation?

   (II) Write down the difference between primitive Pythagorean triple and Pythagorean triple.

Q.5(A) Answer the following questions. (Any Two) **(10)**

1. If $< a_0, a_1, \ldots, a_j > = < b_0, b_1, \ldots, b_n >$ where these finite continued fractions are simple, and if $a_j > 1, and\ b_n > 1$, then

   j = n and $a_i = b_i$ for all $i = 1,2,3, \ldots, n$.
2. Find first three positive solutions of $x^2 - 2y^2 = 1$.
3. If (x, y, z) is Pythagorean triple then

   gcd (x, y, z) = gcd (x, y) = gcd (z, y) = gcd (x, z).

Q.5(B) Answer the following question. (Any One) **(04)**

1. Prove that for all $n \geq 1$,

   (I) $h_n k_{n-1} - h_{n-1} k_n = (-1)^{n-1}$ **(II)** $h_n k_{n-2} - h_{n-2} k_n = (-1)^n a_n$.
2. Find the value of $< 2,1,2,1,2,1, \ldots >$

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